# Two-Level CES Production Technology in the Solow and Diamond Growth Models*

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**Abstract**

The two-level CES aggregate production function - that nests a CES into another CES function - has recently been used extensively in theoretical and empirical applications of macroeconomics. We examine the theoretical properties of this production technology and establish existence and stability conditions of steady states under the Solow and Diamond growth models. It is shown that in the Solow model the sufficient condition for a steady state is fulfilled for a wide range of substitution parameter values. This is in sharp contrast with the two-factor Solow model, where only an elasticity of substitution equal to one is sufficient to guarantee the existence of a steady state. In the Diamond model, multiple equilibria can occur when the aggregate elasticity of substitution is lower than the capital share. Moreover, it is shown that for high initial levels of capital and factor substitutability, the effect of a further increase in a substitution parameter on the steady state depends on capital-skill complementarity.

**Keywords:** Two-level CES production functions, normalization, Solow model, Diamond model, economic growth.

**JEL classification:** *E13, E23, O40, O47.*

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# I. Introduction

The two-factor Cobb-Douglas (CD) aggregate production function has been the preferred specification in growth theory since the seminal contribution of Solow (1956). CD's popularity is hardly surprising given its nice analytical features and its tractability. Nevertheless, some researchers have expressed concerns about the empirical relevance of this specification. Among others it is the only linearly homogenous production function with a constant elasticity of substitution in which each factor share of income is constant over time.[1] More than forty years later, Duffy and Papageorgiou (2000) formally questioned the empirical relevance of CD. Using a panel of 82 countries over a 28 year period they estimated a two-input constant elasticity of substitution (CES) production function specification of which CD is a special case. For the entire sample of countries they strongly rejected the CD specification over the more flexible CES specification. Since then the economic growth literature has been actively moving away from the straight-jacket CD specification to more flexible production functions.[2]

More recently, there is revived interest in a two-level, three-input CES production function that nests one CES function into another. The two-level "nested" CES production technology was pioneered by Sato (1967). Its flexibility, coming from the substitution parameters and the inclusion of an additional input, makes it an attractive choice for many applications in economic theory and empirics. It allows in particular to account for fundamentally different roles of skilled and unskilled labor in the process of economic growth. Griliches developed the notion of capital-skill complementarity (1969) that captures an important aspect of this difference. In recent years, the two-level CES function has been used in a number of empirical studies in order to test for capital-skill complementarity as a cause of the rising wage differential between skilled and unskilled workers.

[1] While Solow (1956) was perhaps the first to suggest the use of the CD specification to characterize aggregate growth, he also noted that there was little evidence to support the choice of such a specification.

[2] A recent collection of empirical and theoretical papers on the CES specification is forthcoming in a special issue of the *Journal of Macroeconomics* March, 2008) entitled "The CES Production Function in the Theory and Empirics of Economic Growth."

Krusell et al. (2000) estimate a variant of the two-level CES function with exogenous technical progress for the U.S. between 1962 and 1993 that explains the wage differential between skilled and unskilled labor. Lindquist (2005) applies a similar approach to Swedish data. Duffy, Papageorgiou and Perez-Sebastian (2004) use the two-level CES production function to test Griliches' capital-skill complementarity hypothesis in a cross-country setting. Using panel data and four different definitions of skilled vs. unskilled workers they find weak evidence in favor of the hypothesis. Employing an alternative dataset and methodology Chmelarova and Papageorgiou (2005) show much stronger evidence for the capital-skill complementarity hypothesis, especially for middle income countries.

In the theoretical literature, the two-level CES productions specification has been used in models of capital-skill complementarity or biased technical change. For example, Goldin and Katz (1998) model the transition between four technologies. Each is characterized by a nested Leontief-Cobb-Douglas aggregate production function that is a special case of the two-level CES function. Acemoglu (1998) considers skill-biased technical progress that enhances the efficiency of skilled labor. Finally, Caselli and Coleman (2006) use the two-level CES production technology in a model of the world technology frontier.

While the two-level CES function with different substitution parameters for skilled and unskilled labor has been used in many studies to account for effects of growth on income distribution, little has been done in examining the effect these parameters have on growth itself. Given the recent empirical support for this function it is important to study how it affects the main implications of basic growth models. In this paper we fill this gap by systematically exploring the properties of the two-level CES function and its effect on the Solow (1956) and Diamond (1965) growth models, the basic workhorses of growth theory. We establish existence and stability conditions of equilibria under these two growth models. Moreover, we examine how changes in the input-substitution parameters underlying capital-skill complementarity can affect transitional growth and the steady state. Our examination draws on recent theoretical contributions by Klump and de La Grandville (2000), Miyagiwa and Papageorgiou (2003), and Karagiannis and Palivos (2007), which examine the effect of the elasticity of substitution between capital and labor on growth under the Solow and Diamond models with two inputs. Following the majority of the existing literature (see, e.g. Klump and de La Grandville (2000); Turnovsky (2002)), we consider changes in the substitution parameters as exogenous.

We obtain the following results: For a given fraction of unskilled labor we can express the substitutability between total labor (skilled and unskilled) and capital by a single aggregate elasticity of substitution. This aggregate elasticity of substitution changes with capital accumulation. In the Solow model, the sufficient condition for a steady state is fulfilled for a wide range of substitution parameters. This sharply contrasts with the two-factor Solow model, where only an elasticity of substitution equal to one (which implies the CD production function) is by itself sufficient to guarantee the existence of a steady state. In addition, we show that an increase in substitution parameters has a positive effect on transitional growth and the steady state. In the Diamond model, we find that multiple equilibria can occur when the elasticity of substitution is lower than the capital share. In addition, we show that the possibility for an inverse relationship between the substitution parameters and growth known from the two-factor case persists in the three-factor nested CES case. But for high capital stocks it now depends on whether capital is more complementary to skilled or to unskilled labor. Finally we consider the effects of an exogenous increase in the fraction of skilled labor and show that it reinforces the effects of higher substitutability of unskilled labor.

The paper is organized as follows. In the next section we introduce the two-level CES function in detail, extending the Klump-de La Grandville normalization of CES functions to this case. We also discuss empirical evidence on substitution parameters and the behavior of the aggregate elasticity of substitution between capital and labor. In sections 3 and 4 we examine the effect of input-substitution parameters of the two-level CES function under the Solow and Diamond growth models. Section 5 considers the

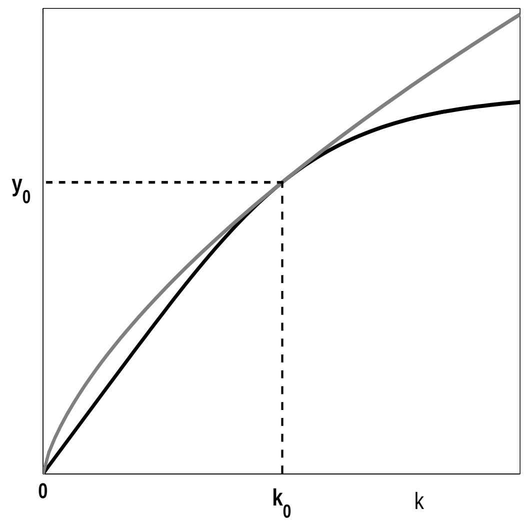


Figure 1: CES functions with a common baseline point

effects of an exogenous increase in the fraction of skilled labor. Section 6 concludes.

## II. Building Blocks

*Klump-de La Grandville CES Normalization*

Production functions with two inputs, constant returns to scale and a constant elasticity of substitution between capital and labor are characterized by three parameters: an efficiency parameter, a distribution parameter (or alternatively two non-neutral efficiency parameters) and a substitution parameter. The substitution parameter determines the curvature of the isoquant. As shown by Klump and de La Grandville (2000, henceforth KL), normalization makes it possible to “straighten” the isoquant in an arbitrary point without shifting it, while holding the efficiency and the distribution parameters constant. This means that any point on a CES function can be chosen as the baseline value for a family of functions that are tangent to it (see the $k$-$y$-diagram of Figure 1). More precisely, a family of normalized CES functions is defined by baseline levels of per capita capital, $k_0$, per capita output, $y_0$, and wage to the interest rate ratio, $\mu_0 = w_0/r_0$. The functions belonging to any family differ only in the elasticities of substitution $\sigma$, where $\sigma = 1/(1-\psi)$ and $-\infty \leq \psi \leq 1$.

In this paper we focus on a two-level, three-factor production technology with capital ($K$), skilled labor ($L_s$) and unskilled labor ($L_u$) as inputs. The first level of the two-level function is given by a CES function

$$X = B[\beta K^{\theta} + (1-\beta)L_s^{\theta}]^{\frac{1}{\theta}}. \tag{1}$$

This CES function is then nested into another CES function, representing the second level given by

$$Y = A[\alpha X^{\psi} + (1-\alpha)L_u^{\psi}]^{\frac{1}{\psi}}. \tag{2}$$

Substituting (1) into (2) yields the two-level CES function

$$Y = A[\alpha B^{\psi}(\beta K^{\theta} + (1-\beta)L_s^{\theta})^{\frac{\psi}{\theta}} + (1-\alpha)L_u^{\psi}]^{\frac{1}{\psi}}. \tag{3}$$

There are two points worth making about equation (3). First, in addition to the efficiency parameter $A$, which is standard to the nested CES functions used in the literature, our formulation includes a second efficiency parameter, $B$. In this way we nest two CES functions with the same structure and refrain from restricting $B$ to be equal to one. Second, out of the multitude of production functions with three factors that have been formulated, the function (3) has proven most suitable for studying growth and income distribution with capital, skilled and unskilled labor as inputs in production. Since Griliches' (1969) contribution there has been an interest in the capital-skill hypothesis for explaining the rising wage inequality between skilled and unskilled workers. Capital-skill complementarity implies that the substitutability between capital and skilled labor is smaller than the substitutability between capital and unskilled labor. To examine this hypothesis, one needs a functional form that is sufficiently general to permit testing for the relevant substitution parameters. A good candidate for this is the nested CES function that has two constant substitution parameters, $\theta$ and $\psi$. With three factors of production there are three possible variants of nesting. Guided by empirical evidence we introduce variant (3) into growth models.[3] Various recent theory-based

[3] One of the other two variants restricts the complementarity between capital and both kinds of

contributions use this nesting as well (see, e.g. Lindquist 2004; Dupuy and de Grip, 2006; Maliar and Maliar, 2006). The capital-skill complementarity hypothesis can also be tested by more flexible functional forms such as the translog function. But they are quite intractable in theoretical analysis.

Next, we apply the KL normalization to the two-level nested CES function. The baseline point is defined by a set of baseline values given by $\{Y_0, X_0, K_0, L_{u0}, L_{s0}\}$. The population is composed of skilled and unskilled workers ($N = L_s + L_u$, $N_0 = L_{s0} + L_{u0}$, $u = L_u/N$). The baseline values in intensive form are given by: $\tilde{y}_0 = Y_0/L_{u0}$, $\tilde{x}_0 = X_0/L_{u0}$, $\hat{x}_0 = X_0/L_{s0}$, $\hat{k}_0 = K_0/L_{s0}$, $y_0 = Y_0/N_0$, $x_0 = X_0/N_0$ and $k_0 = K_0/N_0$. At a constant fraction of skilled labor and for given $N_0$, $x_0$ can be written as a function of $k_0$ only and $k > k_0$ ($k < k_0$) corresponds to $x > x_0$ ($x < x_0$). Lowercase variables designate per capita variables, a tilde denotes values per unskilled worker and a hat denotes values per skilled worker.

Without loss of generality the capital-skill aggregate $X$ is modeled as an input for final output $Y$. The factor prices of capital, skilled and unskilled labor are $r$, $w_s$ and $w_u$. The price of a unit of aggregate $X$ in terms of output is $p_X$. Moreover, we define the income shares $\pi_X = p_X\tilde{x}/\tilde{y}$ and $\pi_K = r\hat{k}/\hat{x}$. The baseline values of relative factor prices are $\mu_0$ and $\nu_0$.

The normalization of the parameters of the production function is obtained from the following conditions:

$$\frac{w_{u0}}{p_{X0}} = \mu_0 = \frac{1-\alpha}{\alpha}\tilde{x}_0^{1-\psi}, \tag{4}$$

$$\frac{w_{s0}}{r_0} = \nu_0 = \frac{1-\beta}{\beta}\hat{k}_0^{1-\theta}, \tag{5}$$

$$X_0 = B[\beta K_0^{\theta} + (1-\beta)L_{s0}^{\theta}]^{\frac{1}{\theta}}, \tag{6}$$

labor to be equal. The third variant restricts the complementary of capital and skilled labor and the complementarity between unskilled and skilled labor to be equal. Intuitively, capital and skilled labor can both be expected to be good substitutes for unskilled labor in the course of economic growth, while capital and skilled labor are expected to be less substitutable. Indeed empirical testing by Fallon and Layard (1975) and Duffy et al. (2004) confirms that these two variants often yield insignificant estimates of negative elasticities of substitution. For a more detailed discussion on the choice of the nested CES function see Fallon and Layard (1975, Appendix A) and Duffy et al. (2004, pp. 333).

$$Y_0 = A[\alpha X_0^{\psi} + (1-\alpha)L_{u0}^{\psi}]^{\frac{1}{\psi}}. \tag{7}$$

The normalized parameters are

$$\alpha = \frac{\tilde{x}_0^{1-\psi}}{\tilde{x}_0^{1-\psi} + \mu_0}, \tag{8}$$

$$\beta = \frac{\hat{k}_0^{1-\theta}}{\hat{k}_0^{1-\theta} + \nu_0}, \tag{9}$$

$$B = \hat{x}_0 \left( \frac{\hat{k}_0^{1-\theta} + \nu_0}{\hat{k}_0 + \nu_0} \right)^{\frac{1}{\theta}}, \tag{10}$$

$$A = \tilde{y}_0 \left( \frac{\tilde{x}_0^{1-\psi} + \mu_0}{\tilde{x}_0 + \mu_0} \right)^{\frac{1}{\psi}}. \tag{11}$$

The parameters of each CES function with two arguments depend only on their own baseline values and their substitution parameters. This is a consequence of the strong separability of the nested CES function (see Sato, 1967). A summary of KL results for normalized CES functions that we use is given in the Appendix (equations (A1)-(A5)).

*Empirical Evidence on Substitution Parameters*

The two-level CES function is frequently used to model and test capital-skill complementarity. Capital-skill complementarity is present if the wage premium for skilled workers increases with capital accumulation. With the two-level CES function specified in (3) this is the case if $\psi > \theta$ (see Krusell et al., 2000). Although the two-level CES function has convenient properties for theoretical applications, obtaining reliable estimates for the substitution parameters $\theta$ and $\psi$ proves to be quite challenging (see Duffy et al., 2004). This is primarily because the substitution parameters depend crucially on the curvature (not the slope) of the production function thus making it difficult to accurately estimate.

The theoretical results presented in this paper can serve as a basis for characterizing different regimes of growth, distribution and factor substitution by different values of the substitution parameters. However, empirical evidence, especially for developing countries, is as yet too scarce and the methods of estimation employed are too diverse to identify such regimes in an empirical way. Nevertheless a few remarks about plausible parameter constellations can be made. The most robust result is that time-series estimations for developed countries in the late 20th century usually find that $\psi > 0 > \theta$, and such values are used for calibration (e.g. Lindquist, 2004 and 2005). Using U.S. manufacturing data, Goldin and Katz (1998) find evidence of increasing capital-skill complementarity in the early 20th century and conjecture that before that time manufacturing was characterized by complementarity between capital and unskilled labor. As the overall substitutability is thought to increase with economic development (see Klump and de La Grandville (2000); Miyagiwa and Papageorgiou (2007)), one would guess that in the 19th century the parameter constellation was $0 > \theta > \psi$, however, there is no empirical evidence to support this hypothesis.

Using international data, Duffy et al. (2004) find panel evidence of capital-skill complementarity and both substitution parameters of the nested CES function being positive. In a nonparametric estimation, Henderson (2005) finds general support for capital-skill complementarity but also several specifications in which complementarity of capital and unskilled labor arises. As far as his results can be assumed not to change too much using a two-level CES function, they would offer strong support for $\psi$ being positive while they would imply positive $\theta$ in several cases.

In sum, we argue that $\psi > 0 > \theta$ is a relevant case for developed countries. Panel estimation seems to offer more support for both substitution parameters being positive. Once again skilled labor appears to be more likely to be complementary to capital than unskilled labor. But overall evidence that would allow us to take a stance on parameter constellations in different countries is still lacking. Given that more and better input data across countries and sectors become available, estimating the two-level CES production function will remain a valuable topic of empirical research on different regimes of economic growth and income distribution.

*Aggregate Elasticity of Substitution*

In view of the many empirical studies investigating capital-skill complementarity, it is important to note that our results on transitional growth and steady states will depend less on capital-skill complementarity than on what we call the *aggregate elasticity of substitution*. Instead of comparing the substitution parameters of both kinds of labor, this elasticity aggregates them into a single value. More specifically, for a constant fraction of unskilled labor $u$ we aggregate skilled and unskilled labor to the total number of workers. We then compute the aggregate elasticity of substitution between capital and the number of workers from the ratio of the average wage rate to the interest rate. It corresponds to the usual elasticity of substitution of a function with two inputs. While this elasticity is constant for the basic CES function, it is variable for the two-level CES function. The average wage rate $w$ is obtained by multiplying the total wage share by the income per person: $w = (1 - \pi_X \pi_K)y$.

The following lemmas formally describe two relevant properties for our subsequent investigation of the Solow and Diamond growth models. All proofs are given in the Appendix:

**Lemma 1.** *For a constant fraction of unskilled workers $u$ the elasticity of substitution between capital $K$ and the number of workers $N$ in the two-level CES function (3) is a harmonic mean of the two-factor elasticities within each CES function, $1/(1-\psi)$ and $1/(1-\theta)$:*

$$\sigma = \frac{w/r}{k\frac{\partial w/r}{\partial k}} = \left[(1-\theta)\frac{1-\pi_K}{1-\pi_X\pi_K} + (1-\psi)\frac{\pi_K(1-\pi_X)}{1-\pi_X\pi_K}\right]^{-1}$$
$$= \left[(1-\theta)(1-g) + (1-\psi)g\right]^{-1},$$

where $g = \frac{\pi_K(1-\pi_X)}{1-\pi_X\pi_K}$.

**Lemma 2.**

*(i) If $\theta$ and $\psi$ have opposing signs or if $|\theta| > |\psi|$, $\lim_{k\to 0}\sigma = \max[1/(1-\theta), 1/(1-\psi)]$. and $\lim_{k\to\infty}\sigma = \min[\frac{1}{1-\theta}, \frac{1}{1-\psi}]$.*

*(ii) If $\psi > \theta > 0$ or $0 > \theta > \psi$, both limits are equal to $1/1-\theta$.*

We note that the aggregate elasticity of substitution is declining in $k$ when the substitution parameters have opposing signs, which is an empirically relevant case. Other theoretical and empirical studies investigating a production function with two factors point to an elasticity of substitution that increases with capital accumulation. But keeping in mind that we assume a constant technology and constant skill-levels, the result of a decreasing elasticity of substitution appears less surprising. Hicks (1963, p. 132) already speculated in favor of a declining elasticity of substitution that "... may be counteracted by invention."

# III. The Solow Model

*Existence and Stability of Steady States*

Next we introduce the two-level CES function into the basic Solow model. We assume that total population, skilled and unskilled labor all grow at the same rate $n$, which leaves the fraction of unskilled labor $u$ constant. Also, for simplicity of exposition we assume $\theta \neq \psi$ and $\theta, \psi \neq 0$. The savings ratio $s$ is exogenous as well as the depreciation rate $\delta$.

Capital accumulates according to the standard motion equation

$$\dot{k} = sy - (n + \delta)k. \tag{12}$$

The steady state condition then easily derives as

$$\begin{aligned} sy^* &= (n+\delta)k^* \\ \Leftrightarrow sA[\alpha B^{\psi}(\beta k^{*\theta} + (1-\beta)(1-u)^{\theta})^{\frac{\psi}{\theta}} + (1-\alpha)u^{\psi}]^{\frac{1}{\psi}} &= (n+\delta)k^*, \end{aligned} \tag{13}$$

where (*) denotes steady-state values. As with two inputs, the economy can experience continuous decline, converge to a constant steady state or grow endogenously in the long-run.

**Proposition 1.** *Under the Solow model with the two-level CES function (3) the following holds:*

*(i) For* $\psi$ *and* $\theta$ *both positive, a steady state* $k^* > 0$ *exists if and only if* $A\alpha^{1/\psi}B\beta^{1/\theta} \leq (n+\delta)/s$*, otherwise* $k^* \to \infty$.

*(ii) For* $\psi$ *and* $\theta$ *both negative, a steady state* $k^* > 0$ *exists if and only if* $A\alpha^{1/\psi}B\beta^{1/\theta} > (n+\delta)/s$*, otherwise* $k^* = 0$.

*(iii) If* $\theta$ *and* $\psi$ *have the opposite sign, a steady state* $k^* > 0$ *always exists.*

*(iv) A positive and finite steady state is always unique and stable.*

In cases *(i)* and *(ii)* in which both substitution parameters $\psi$ *and* $\theta$ are positive or negative, the results for the two-level CES function correspond to the results for the two-input CES function in Klump and Preissler (2000). The result on endogenous growth for case *(i)* also follows from studying the limiting behavior of the aggregate elasticity of substitution (see, Karagiannis and Palivos, 2007). A notable difference from the two-input CES case arises when the parameters have opposing signs. That is, in the Solow model with the two-input CES function, the only substitution parameter value sufficient for a finite positive steady state is zero which of course implies the CD function. Proposition 1 *(iii)* shows that under the two-level three-input CES function, the sufficient condition for a finite positive steady state does not require the knife-edge assumption required in the two-input CES case. Whenever the substitution parameters have opposing signs such a steady state exists.

The results can be explained from the properties of the basic two-input CES function. For the latter, the marginal product of capital becomes infinite when $k$ converges to zero only if the substitution parameter is nonnegative, otherwise it converges to a finite level. For $k$ going to infinity, the marginal product converges to zero only if the substitution parameter is not positive, otherwise it converges to a positive finite level. In the two-level CES function an infinite marginal product at one level for $k$ going to zero is sufficient to make the overall marginal product converge to infinity. Similarly, a marginal product converging to zero at one level for $k$ going to infinity is sufficient to make the overall marginal product converge to zero (see Figures 2-4).

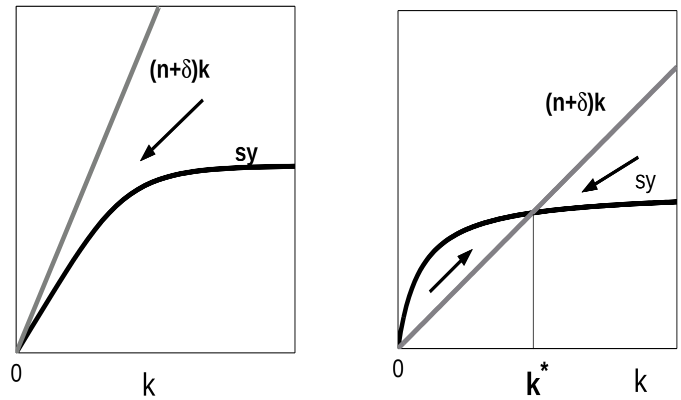


Figure 2: Steady state in Solow model with both $\psi$ and $\theta < 0$

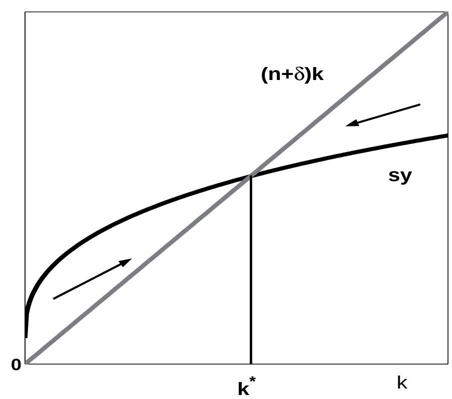


Figure 3: Steady state in Solow model with $\psi$ and $\theta$ having opposing signs

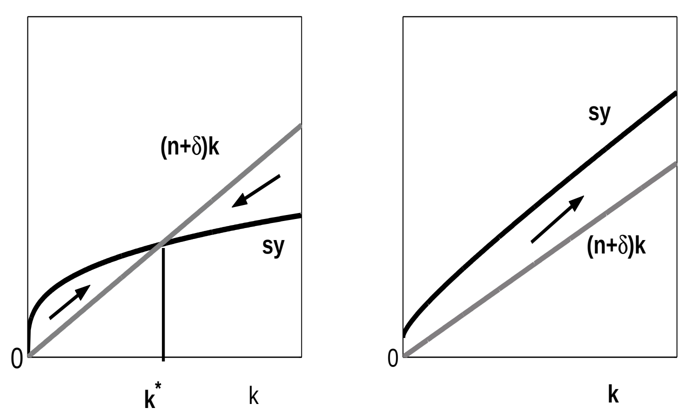


Figure 4: Steady state in Solow model with both $\psi$ and $\theta > 0$

As we have mentioned, $\psi > 0$ is empirically plausible. This implies that a positive steady state exists. It is less clear what determines the magnitude of $\theta$. In an economy with $\theta < 0$ factor substitution cannot be a source of long-run growth.

*Effects of Substitution Parameters on Steady States*

In the Solow model, savings are a constant fraction of average income. The effect of higher substitutability on transitional growth and the steady state derives from its effect on average income:

**Proposition 2.** *At given input quantities with $k \neq k_0$, an increase in any substitution parameter in the two-level CES function (3) has a positive effect on output per capita.*

This proposition is consistent with KL theorems for the two-factor case. Notice that it is independent of any model assumption and therefore holds for both the Solow and Diamond growth models. The more substitutable inputs are, the less marginal returns of the factor that is used at a rising intensity decline. This translates into higher average productivity. In the Solow model, the positive effect for $k \neq k_0$ carries over to transitional growth and the steady state. The steady state effect is easily obtained applying the implicit function theorem.

Although the result is quite straightforward it points to an intriguing concept: namely that substitution parameters (in our case $\theta$ and $\psi$) can be thought of as alternative sources of technological process that work in similar vain as the efficiency parameter $A$. While not the focus of this paper, investigating empirically and theoretically the behavior of such parameters is an important step worthy of future research.[4]

[4] For example, Miyagiwa and Papageorgiou (2007) present one possibility in which the elasticity of substitution between capital and labor can be endogenized. This work can certainly be extended in various directions using a three-input two-level CES specification. See also Benabou (2005).

# IV. The Diamond Model

Given that one of the main motivations for choosing to work with the two-level CES function instead of the basic CES function is to account for changes in the distribution of income due to capital accumulation, we extend our analysis to another highly influential growth model, the Diamond (1965) model, in which distribution affects growth. In the Diamond model with logarithmic utility, savings turn out to be a constant fraction of wage income. We restrict our attention to this case. As in the Solow model, we assume $\theta \neq \psi$ and $\theta, \psi \neq 0$ as well as constant population growth and a constant fraction of unskilled labor.

In the Diamond model with two inputs, the effects of the elasticity of substitution on transitional growth and the steady state differ from the Solow model in two ways. First, the elasticity of substitution affects the uniqueness and stability of steady states. Second, a higher elasticity of substitution does not always have a positive effect on transitional growth and the steady state. We show to what extent the results are carried over to the model with skilled and unskilled labor. The reason for the differences to the Solow model is that growth now depends on the distribution of income between capital, skilled labor and unskilled labor.

*Existence and Stability of Steady States*

The equation of capital accumulation in period $t$ is given by

$$k_{t+1} = sy_t(1 - \pi_{Xt} + \pi_{Xt}(1 - \pi_{Kt})) = sw_t, \tag{14}$$

and the steady state condition is given by

$$sw^* = k^*, \tag{15}$$

where $s$ is now the savings ratio out of wages only.

The stability of a steady state hinges on the derivative $\partial w/\partial k$.

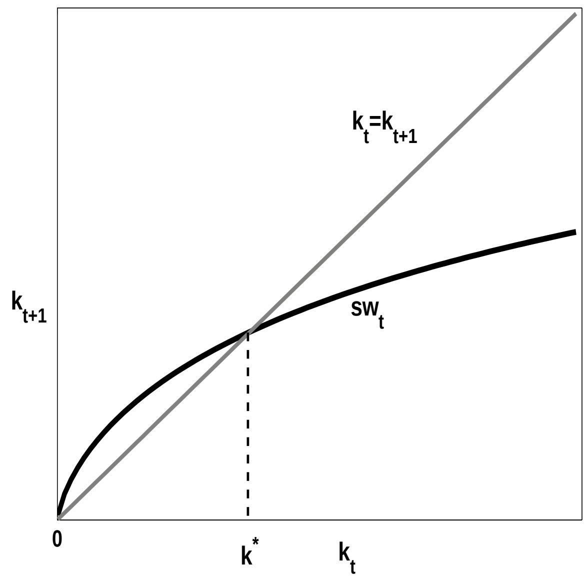


Figure 5: $\sigma \geq \pi_X \pi_K$, a unique and stable steady state exists in the Diamond model

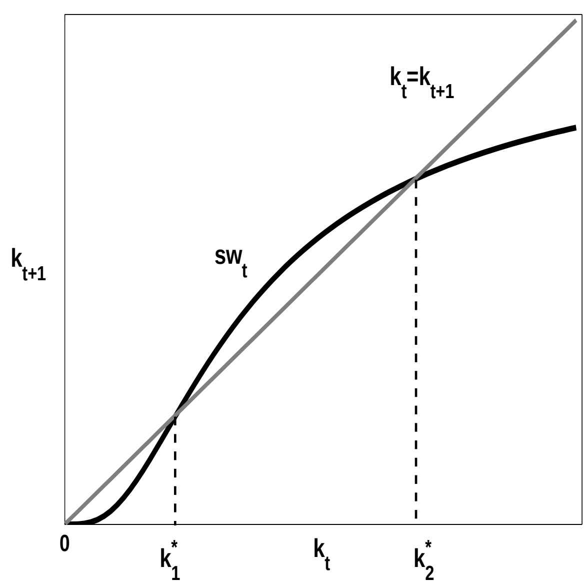


Figure 6: For $\sigma < \pi_X \pi_K$, unstable steady states are possible

**Proposition 3.** *In the Diamond model with constant savings out of wages and the two-level CES function (3) the following holds:*

*(i) If $\theta$ and $\psi$ are both positive, exactly one positive steady state exists and is stable.*

*(ii) If $\theta$ and $\psi$ have the opposite sign, at least one positive steady state exists. In case of multiple steady states, the lowest and the highest steady state are stable.*

*(iii) If $\psi$ and $\theta$ are both negative, there exist either multiple positive steady states or none (except for $s\partial w/\partial k$ only once tangent to $k_t = k_{t+1}$). In the case of multiple steady states the lowest is unstable and the highest is stable.*

*(iv) Unstable steady states only occur if the aggregate elasticity of substitution $\sigma$ is lower than the capital share $\pi_X \pi_K$.*

As $\pi_X \pi_K$ and $\sigma < 1$ decline jointly in most cases, it is difficult to exclude more than one unstable equilibrium analytically. But with plausible parameter values the condition for instability will only be fulfilled for a small range of the capital stock. Unstable equilibria become impossible as soon as the capital share has fallen below the lower bound of the elasticity of substitution. In simulations we have never found more than one unstable equilibrium. Thus the simulations indicate that there is a single and stable equilibrium in cases *(i)* and *(ii)* and that there are no or two equilibria in case *(iii)*. As in the two-factor model (see Azariadis, 1996; Miyagiwa and Papageorgiou, 2003) unstable equilibria only occur for an aggregate elasticity of substitution below one. Moreover, they do not depend on capital-skill complementarity. Rather they depend on the ratio of the output elasticity of capital to the aggregate elasticity of substitution. This ratio determines the curvature of the savings function.

If $\psi > 0$ we are in a case for which we have not found any unstable equilibria in simulations. However, the robustness of this parameter value remains to be checked for developing countries.

*Effects of Substitution Parameters on Steady States*

Klump and de La Grandville (2000) consider the elasticity of substitution between capital and labor as an engine of growth. As it always has a positive effect on income per capita for $k \neq k_0$, the argument is not tied to the Solow model. It does not remain valid without qualification, however, as soon as savings are not made out of average income but depend on its distribution. With two factors of production, Miyagiwa and Papageorgiou (2003) and independently Irmen (2001) have shown that the elasticity of substitution has a threshold above one for which its effect on wages is always negative. It is moreover possible to show that irrespective of $\sigma$ the effect is negative for a certain range of $k$ with $k > k_0$. Assuming $k > k_0$ there are thus two reasons for a negative effect of a rise in the elasticity of substitution on transitional growth and the steady state: a too low capital stock or a too high initial elasticity of substitution. We examine how this result changes when the two-level CES function is introduced.

With a fixed savings ratio, capital accumulation depends only on the average wage $w$. Given a capital stock $k_t > k_0$ the influence of a higher substitution parameter $\psi$ on next period's capital stock $k_{t+1}$ is

$$\frac{\partial k_{t+1}}{\partial \psi} = s\frac{\partial w_t}{\partial \psi}, \tag{16}$$

and analogously for $\theta$

$$\frac{\partial k_{t+1}}{\partial \theta} = s\frac{\partial w_t}{\partial \theta}. \tag{17}$$

Omitting the time subscript and using (14) we obtain

$$\frac{\partial w}{\partial \psi} = \frac{\partial y}{\partial \psi}(1 - \pi_X \pi_K) - y\pi_K \frac{\partial \pi_X}{\partial \psi}, \tag{18}$$

and

$$\frac{\partial w}{\partial \theta} = \frac{\partial y}{\partial \theta}(1 - \pi_X \pi_K) - y\pi_X \frac{\partial \pi_K}{\partial \theta} - y\pi_K \frac{\partial \pi_X}{\partial \theta}. \tag{19}$$

**Proposition 4.** *In the Diamond model with constant savings out of wages and the two-level CES function (3) the following holds:*

*(i)* $\partial k_{t+1}/\partial\psi$ *and* $\partial k_{t+1}/\partial\theta$ *are positive for* $k < k_0$*, zero for* $k = k_0$ *and negative in an interval* $(k_0, k_0 + \epsilon]$*,* $\epsilon$ *being an arbitrarily small positive number.*

*(ii)* $\partial k_{t+1}/\partial\psi$ *is positive for* $k \to \infty$ *if at least one substitution parameter is negative or if* $\psi > \theta$*. It is negative if* $\theta > \psi > 0$.

*(iii)* $\partial k_{t+1}/\partial\theta$ *is positive for* $k \to \infty$ *if at least one substitution parameter is negative or if* $\theta > \psi$*. It is negative if* $\psi > \theta > 0$.

We are not able to exclude multiple changes in the sign of the derivatives for $k > k_0$ analytically. But in numerous simulations we have obtained only one change. These results carry over to stable steady states by a straightforward application of the implicit function theorem. The effect on unstable steady states is always opposite.

Klump and Saam (forthcoming) explain that with $k < k_0$ there would be unemployment of labor if the elasticity of substitution fell to zero, while there would be unemployment of capital with $k > k_0$. Thus with $k > k_0$ a positive elasticity of substitution enables capital accumulation above $k_0$. They argue this case corresponds to the situation simple growth models are concerned with. We therefore consider $k > k_0$ as the empirically relevant range of the capital stock. In this case, a negative effect of substitution parameters on the average wage can only occur for low levels of the capital stock.

We see that for positive substitution parameters the effect of a further rise now depends on the initial capital stock as well as on the two parameters underlying capital-skill complementarity. The reason lies in the fact that these three magnitudes together determine the evolution of wages. Even if the rise in $\psi$ lowers the wage of unskilled workers, the average wage may increase if the rise in the wage of skilled workers is sufficiently strong. The same applies to a rise in $\theta$ if it induces a fall in the wage of skilled workers. As discussed in section 2.2, there is to date no strong evidence for cases in which both parameters are positive but that possibility cannot be rejected either.

# V. Changes in the Fraction of Skilled Labor

So far we have assumed that the fraction of skilled labor in the total workforce is constant. But generally the fraction of skilled labor tends to rise with economic development. In the model, the rise can be one-time or continuous and it can be exogenous or endogenous. In this paper we take a look at a one-time exogenous change, leaving other possibilities to future research.[5]

It is plausible to consider that the fraction of skilled labor will only rise as long as the wage for skilled workers is higher than for unskilled workers. In this situation, a rise in the share of skilled workers will always increase output per worker, workers' marginal productivity in their new job, $w_S$, being higher than the marginal productivity in their old job, $w_U$. From the proof of Proposition 1 we see that while having an effect on output, the fraction of skilled labor does not affect the existence of a steady state or a positive long-run growth rate in the Solow model.

We now examine whether a rise in the fraction of skilled labor benefits more to an economy with high factor substitutability or with low factor substitutability. The cross-derivative that evaluates this effect can also be read inversely: It shows how an increase in productivity caused by higher factor substitutability depends on the fraction of skilled labor.

The relative increase in income brought about by a rise in the fraction of skilled labor $(1-u)$ is:

$$-\frac{\frac{\partial y}{\partial u}}{y} = \frac{w_S - w_U}{y} = \frac{\pi_X(1-\pi_K u) - (1-u)}{(1-u)u}. \tag{20}$$

Taking the derivatives with respect to the substitution parameters yields

$$\frac{\partial\left(\frac{w_S - w_U}{y}\right)}{\partial\psi} = \frac{1-\pi_K u}{u(1-u)}\frac{\partial\pi_X}{\partial\psi}, \tag{21}$$

[5] Maliar and Maliar (2006) introduce different constant growth rates of skilled and unskilled labor. In order for a steady state in the usual sense to exist, they have to be offset by different rates of labor-biased technical change.

$$\frac{\partial\left(\frac{w_S - w_U}{y}\right)}{\partial\theta} = \frac{1-\pi_K u}{u(1-u)}\frac{\partial\pi_X}{\partial x}\frac{\partial x}{\partial\theta} - \frac{\pi_X}{1-u}\frac{\partial\pi_K}{\partial\theta}. \tag{22}$$

Proposition 5 immediately follows using(A3) to (A5):

**Proposition 5.**

*(i) For $k > k_0$, the growth in output $y$ caused by a marginal increase in the fraction of skilled labor $(1-u)$ depends positively on $\psi$.*

*(ii) For $k > k_0$ and $\psi < 0$, the growth in output $y$ caused by a marginal increase in the fraction of skilled labor $(1-u)$ depends negatively on $\theta$.*

The converse is true for $k < k_0$. Considering $k > k_0$ as the relevant situation, there is always a positive interaction in the effect of a higher substitution parameter $\psi$ and a higher fraction of skilled labor $1-u$ on output. Higher substitutability between the capital-skill aggregate and unskilled labor raises the fraction of skilled labor and thus its productivity in relation to output.

The interaction in the effect of a higher substitution parameter $\theta$ and a higher fraction of skilled labor $1-u$ on output is negative if $k > k_0$ and $\psi < 0$, as $\pi_X$ is declining is $x$ in this case. For $\psi > 0$ the effect is ambiguous. The total income share of capital and skilled labor rises with $\theta$, but the fraction of skilled labor relative to capital declines.

## VI. Conclusion

Motivated by favorable empirical evidence and revived theoretical interest, we considered the Solow and Diamond growth models with a two-level CES production function. Existence and stability conditions for steady states were derived and the effects of the substitution parameters on transitional growth and the steady state were examined. In general, our theoretical results show that the substitution parameters of the two-level CES function, which have so far mostly been considered to characterize distributional effects of growth, are major determinants of growth themselves. While some results can be carried over from the case with two factors to the case with three factors, two notable differences arise. First, within the family of basic two-factor CES production functions, the Cobb-Douglas is the only function for which a steady state is guaranteed independently of the other parameter values. In contrast, we show that with the three-factor two-level CES function examined here, a steady state is guaranteed whenever the parameters have opposite signs. Thus the sufficient condition for the existence of a steady state becomes broader under our nested CES function than the knife-edge assumption of an elasticity of substitution equal to one in the two-factor case. Second, in the Diamond model negative effects of factor substitutability rising to high levels, which were found in the two-factor case, do not occur for large capital stocks as long as only the initially higher of both substitution parameters increases. In other words, in this case the effect of higher substitutability depends on the presence of capital-skill complementarity. An important extension of our analysis is the introduction of changes in the fraction of skilled labor. We made a first step in considering an exogenous one-time change. Under plausible assumptions the effect on output depends positively on the substitution parameter between capital and unskilled labor and in an ambiguous way on the substitution parameter between capital and skilled labor. Future research should consider continuous or endogenous changes in the fraction of skilled labor. Moreover, further empirical research should help identify different regimes of substitution between the three factors across countries and over time.

# A Appendix

*Results on the CES Function with Two Factors*

KL show the following results for the basic CES function. They are written down for both levels of our function:

$$\frac{\tilde{y}}{\tilde{y}_0} = \frac{y}{y_0} = \left(\frac{1-\pi_{X0}}{1-\pi_X}\right)^{\frac{1}{\psi}}, \qquad \frac{\hat{x}}{\hat{x}_0} = \frac{x}{x_0} = \left(\frac{1-\pi_{K0}}{1-\pi_K}\right)^{\frac{1}{\theta}} \tag{A1}$$

$$\frac{\tilde{x}}{\tilde{x}_0} = \frac{x}{x_0} = \left(\frac{\pi_X(1-\pi_{X0})}{\pi_{X0}(1-\pi_X)}\right)^{\frac{1}{\psi}}, \qquad \frac{\hat{k}}{\hat{k}_0} = \frac{k}{k_0} = \left(\frac{\pi_K(1-\pi_{K0})}{\pi_{K0}(1-\pi_K)}\right)^{\frac{1}{\theta}}. \tag{A2}$$

$$\frac{\partial \pi_X}{\partial x} = \frac{\psi}{x}\pi_X(1-\pi_X), \qquad \frac{\partial \pi_K}{\partial k} = \frac{\theta}{k}\pi_K(1-\pi_K) \tag{A3}$$

$$\frac{\partial \pi_X}{\partial \psi} = \pi_X(1-\pi_X)\ln\left(\frac{x}{x_0}\right), \qquad \frac{\partial \pi_K}{\partial \theta} = \pi_K(1-\pi_K)\ln\left(\frac{k}{k_0}\right) \tag{A4}$$

$$\begin{aligned}
\frac{\partial \tilde{y}}{\partial \psi} &= -\frac{\tilde{y}}{\psi^2}\left(\pi_X \ln\left(\frac{\pi_{X0}}{\pi_X}\right) + (1-\pi_X)\ln\left(\frac{1-\pi_{X0}}{1-\pi_X}\right)\right) > 0 \quad \text{for} \quad x \neq x_0 \Leftrightarrow k \neq k_0 \\
\frac{\partial \hat{x}}{\partial \theta} &= -\frac{\hat{x}}{\theta^2}\left(\pi_K \ln\left(\frac{\pi_{K0}}{\pi_K}\right) + (1-\pi_K)\ln\left(\frac{1-\pi_{K0}}{1-\pi_K}\right)\right) > 0 \quad \text{for} \quad k \neq k_0
\end{aligned} \tag{A5}$$

For subsequent proofs we write

$$\begin{aligned}
\pi_X \ln\left(\frac{\pi_{X0}}{\pi_X}\right) + (1-\pi_X)\ln\left(\frac{1-\pi_{X0}}{1-\pi_X}\right) &= \Phi_X < 0 \quad \text{for} \quad k \neq k_0 \\
\pi_K \ln\left(\frac{\pi_{K0}}{\pi_K}\right) + (1-\pi_K)\ln\left(\frac{1-\pi_{K0}}{1-\pi_K}\right) &= \Phi_K < 0 \quad \text{for} \quad k \neq k_0
\end{aligned} \tag{A6}$$

*Proof of Lemma 1*

For taking its derivative, we write the ratio of the wage to the interest rate as

$$\frac{w}{r} = \frac{1-\pi_X\pi_K}{\pi_X\pi_K}k. \tag{A7}$$

We obtain the derivative of the capital share making use of (A3):

$$\begin{aligned}
\frac{\partial \pi_X\pi_K}{\partial k} &= \pi_K\frac{\partial \pi_X}{\partial x}\frac{\partial x}{\partial k} + \pi_X\frac{\partial \pi_K}{\partial k} \\
&= \pi_K\frac{\psi}{x}\pi_X(1-\pi_X)\pi_K\frac{x}{k} + \pi_X\frac{\theta}{k}\pi_K(1-\pi_K) \\
&= \frac{\pi_X\pi_K}{k}(\psi\pi_K(1-\pi_X) + \theta(1-\pi_K)).
\end{aligned} \tag{A8}$$

With (A7) and (A8) we obtain

$$\sigma = \frac{w/r}{k\frac{\partial w/r}{\partial k}} = \frac{1-\pi_X\pi_K}{(1-\psi)\pi_K(1-\pi_X)+(1-\theta)(1-\pi_K)}. \blacksquare \tag{A9}$$

*Proof of Lemma 2*

We rewrite the result from Lemma 1 as

$$\frac{1}{\sigma} = (1-\theta)+(\theta-\psi)g \tag{A10}$$

with

$$g = \frac{\pi_K}{\frac{1-\pi_K}{1-\pi_X}+\pi_K}. \tag{A11}$$

If $\psi$ and $\theta$ have opposing signs, the limits are straightforward. Note that for $\psi > 0 > \theta$ the limit of $\pi_X$ for $k \to \infty$ is lower than one because $x$ is bounded. In this case

$$\lim_{k\to 0} g = 1 \qquad \lim_{k\to\infty} g = 0. \tag{A12}$$

For $\theta > 0 > \psi$:

$$\lim_{k\to 0} g = 0 \qquad \lim_{k\to\infty} g = 1. \tag{A13}$$

If the substitution parameters have the same sign, we evaluate the limit of $\frac{1-\pi_K}{1-\pi_X}$ using (A1) and (A2):

$$\frac{1-\pi_K}{1-\pi_X} = (1-\pi_{K0})\left(\frac{\pi_{X0}}{1-\pi_{X0}}\right)^{\frac{\theta}{\psi}}\left(\frac{1}{\pi_X}\right)^{\frac{\theta}{\psi}}(1-\pi_X)^{\frac{\theta}{\psi}-1} \tag{A14}$$

Plugging the result into (A11) we obtain: For $\psi > \theta > 0$ and $0 > \theta > \psi$

$$\lim_{k\to 0} g = 0 \qquad \lim_{k\to\infty} g = 0, \tag{A15}$$

for $\theta > \psi > 0$

$$\lim_{k\to 0} g = 0 \qquad \lim_{k\to\infty} g = 1, \tag{A16}$$

and for $0 > \psi > \theta$

$$\lim_{k\to 0} g = 1 \qquad \lim_{k\to\infty} g = 0. \tag{A17}$$

Plugging the results (A12)-(A17) into (A10) yields Lemma 2 *(i)* and *(ii)*.■

*Proof of Proposition 1*

For $\theta, \psi > 0$ we show the condition for endogenous growth

$$\lim_{k\to\infty} \frac{\dot{k}}{k} = \lim_{k\to\infty} sA \left[ \alpha B^{\psi} \left( \beta + (1-\beta) \left( \frac{1-u}{k} \right)^{\theta} \right)^{\frac{\psi}{\theta}} + (1-\alpha) \left( \frac{u}{k} \right)^{\psi} \right]^{\frac{1}{\psi}} - (n+\delta) > 0$$
$$\Leftrightarrow A\alpha^{\frac{1}{\psi}} B\beta^{\frac{1}{\theta}} > \frac{n+\delta}{s} \tag{A18}$$

If $\theta, \psi < 0$ the condition for the existence of a positive steady state $k^*$ is

$$\lim_{k\to 0} sA \left( \frac{1}{\alpha B^{\psi} \left( \beta + (1-\beta) \left( \frac{k}{1-u} \right)^{-\theta} \right)^{\frac{\psi}{\theta}} + (1-\alpha) \left( \frac{k}{u} \right)^{-\psi}} \right)^{-\frac{1}{\psi}} - (n+\delta) > 0$$
$$\Leftrightarrow A\alpha^{\frac{1}{\psi}} B\beta^{\frac{1}{\theta}} > \frac{n+\delta}{s}. \tag{A19}$$

If $\psi < 0$ and $\theta > 0$, we see that the condition for endogenous growth is never fulfilled

$$\lim_{k\to\infty} sA \left( \frac{1}{\frac{1}{\alpha B^{\psi} \left( \beta + (1-\beta) \left( \frac{1-u}{k} \right)^{\theta} \right)^{-\frac{\psi}{\theta}}} + (1-\alpha) \left( \frac{k}{u} \right)^{-\psi}} \right)^{-\frac{1}{\psi}} - (n+\delta) = -(n+\delta), \tag{A20}$$

and a steady state $k^*$ always exists

$$\lim_{k\to 0} sA \left( \frac{1}{\frac{1}{\alpha B^{\psi} \left( \beta + (1-\beta) \left( \frac{1-u}{k} \right)^{\theta} \right)^{-\frac{\psi}{\theta}}} + (1-\alpha) \left( \frac{k}{u} \right)^{-\psi}} \right)^{-\frac{1}{\psi}} - (n+\delta) = \infty - (n+\delta) > 0. \tag{A21}$$

In an analogous way it is shown that a steady state always exists for $\psi > 0$ and $\theta < 0$. From (A20) and (A21) follows easily that endogenous growth never occurs if both parameters are negative and that $k^* > 0$ if at least one parameter is positive.

Part *(iv)* follows from the declining marginal product of capital.■

*Proof of Proposition 2*

As $L_u$ and $L_s$ do not depend on $\psi$ and $\theta$, it follows from (A5) that

$$\frac{\partial y}{\partial \psi} > 0, \quad \frac{\partial x}{\partial \theta} > 0 \quad \text{for} \quad k \neq k_0. \tag{A22}$$

The effect of $\theta$ on $y$ is obtained as

$$\frac{\partial y}{\partial \theta} = \frac{\partial y}{\partial x}\frac{\partial x}{\partial \theta} > 0 \quad \text{for} \quad k \neq k_0. \blacksquare \tag{A23}$$

*Proof of Proposition 3*

From the steady state condition (15) follows

$$s(1 - \pi_X \pi_K^*) y^* = k^* \Leftrightarrow \frac{1}{s} = (1 - \pi_X \pi_K^*) \frac{y^*}{k^*} \tag{A24}$$

A steady state is stable if and only if

$$\begin{aligned} \left| s\frac{\partial w}{\partial k} \right|_{k=k^*} &< 1 \\ \Leftrightarrow \left( s\pi_X \pi_K \frac{y}{k} \left[ (1-\psi)(1-\pi_X)\pi_K + (1-\theta)(1-\pi_K) \right] \right)_{|k=k^*} &< 1. \end{aligned} \tag{A25}$$

To obtain (A25) we have written the marginal product of capital as $\partial y/\partial x \cdot \partial x/\partial k = \pi_X \pi_K y/k$ and used (A8).

To proof *(i)-(iii)* we study the limiting behavior of $\partial w/\partial k$. From Proposition 2 follows: If at least one parameter is negative, the marginal product of capital converges to a positive finite value for $k \to 0$ and to zero for $k \to \infty$. If both parameters are positive, it converges to infinity for $k \to 0$ and to a positive finite value for $k \to \infty$. Evaluating the marginal product and the income shares in (A25) we obtain $\lim_{k\to\infty}(s\partial w/\partial k) = 0$ irrespective of $\psi$ and $\theta$. If $\theta, \psi < 0$, $\lim_{k\to 0}(s\partial w/\partial k) = 0$. If at least one parameter is positive, $\lim_{k\to 0}(s\partial w/\partial k) = \infty$. As $\partial w/\partial k$ is continuous, part *(i)-(iii)* follows.■

To show *(iv)* we plug (A24) and the aggregate elasticity of substitution from Lemma 1 into (A25) and obtain the following necessary and sufficient condition for stability:

$$\frac{\pi^*}{\sigma^*} < 1 \Leftrightarrow \pi^* < \sigma^*. \blacksquare \tag{A26}$$

*Proof of Proposition 4*

*Part (i)* Using (A1)-(A5) we rewrite (18) as

$$\lim_{k\to\infty} \frac{\partial w}{\partial \psi} = \lim_{k\to\infty} \left(\frac{-y}{\psi^2}\right) \left[\Phi_X(1-\pi_X\pi_K) + \psi\pi_K\pi_X(1-\pi_X)\ln\left(\frac{\pi_X(1-\pi_{X0})}{\pi_{X0}(1-\pi_X)}\right)\right]. \tag{A27}$$

As $\Phi_x < 0$ for $k \neq k_0$, $\partial w/\partial\psi$ is positive for $k < k_0$. It is obviously zero at $k = k_0$. In order to obtain the sign of $\partial w/\partial\psi$ at $k_0+\epsilon$, $\epsilon$ being an arbitrarily small positive number, we differentiate the expression in brackets with respect to $k$ (see Irmen (2001) for the two-factor case):

$$\begin{aligned}\frac{\partial[...]}{\partial k} &= \frac{\partial \Phi_X}{\partial \pi_X}\frac{\partial \pi_X}{\partial k}\left(\frac{1-\pi_K\pi_X}{\pi_K\pi_X(1-\pi_X)}\right) + \Phi_X\frac{\partial \frac{1-\pi_X\pi_K}{\pi_X\pi_K(1-\pi_X)}}{\partial k} + \frac{\psi^2}{x}\frac{\partial x}{\partial k} \\ &= \ln\left(\frac{\pi_{X0}(1-\pi_X)}{\pi_X(1-\pi_{X0})}\right)\frac{\partial \pi_X}{\partial k}\left(\frac{1-\pi_K\pi_X}{\pi_K\pi_X(1-\pi_X)}\right) + \Phi_X\frac{\partial\left(\frac{1-\pi_X\pi_K}{\pi_X\pi_K}\right)}{\partial k} + \frac{\psi^2}{x}\frac{\partial x}{\partial k}.\end{aligned} \tag{A28}$$

For $k = k_0$ follows

$$\frac{\partial[...]}{\partial k} = 0 + 0 + \frac{\psi^2}{x_0} > 0 \Leftrightarrow \frac{\partial w}{\partial\psi}(k_0+\epsilon) < 0. \tag{A29}$$

The results for a change in $\theta$ are obtained in the same way. Again $\frac{\partial w}{\partial\theta}$ is zero at the baseline point. As the derivatives of $\pi_X$, $\pi_K$ and $\Phi_K$ with respect to $k$ are zero at this point, the cross derivative of $w$ with respect to $\theta$ and $k$ is negative. For $k_0+\epsilon$, $\partial w/\partial\theta$ is thus negative.$\blacksquare$

*Part (ii)*: Using (A1) we rewrite (A27) as

$$\lim_{k\to\infty} \frac{\partial w}{\partial \psi} = \lim_{k\to\infty} \left(\frac{-y}{\psi^2}\right) \left[\Phi_X(1-\pi_X\pi_K) + \psi\pi_K\pi_X(1-\pi_X)\ln\left(\frac{\pi_X(1-\pi_{X0})}{\pi_{X0}(1-\pi_X)}\right)\right]. \tag{A30}$$

For $\theta < 0$ or $\psi < 0$, $y$ has an upper bound and $\pi_X \pi_K$ converges to zero. In functions of the type $z^\eta \ln z$ with $\eta > 0$ the logarithm converges more slowly, in particular $\lim_{z \to 0} z \ln z = 0$. The expression in square brackets converges thus to $\ln \pi_0$ or $\ln(1 - \pi_0)$. If both parameters are positive, the aggregate elasticity of substitution plays a central role through the weight $g$.

For the case that $\psi > \theta > 0$ we rewrite (A27)

$$\lim_{k \to \infty} \frac{\partial w}{\partial \psi} = \lim_{k \to \infty} \left( \frac{-y}{\psi^2} \right) (1 - \pi_X \pi_K) \left[ \Phi_X + g \pi_X \ln \left( \frac{\pi_X (1 - \pi_{X0})}{\pi_{X0}(1 - \pi_X)} \right) \right]. \tag{A31}$$

With (A1) we see that in $y(1 - \pi_X)$ the convergence of $y$ to infinity dominates for $1 > \theta, \psi > 0$. Hence also $y(1 - \pi_X \pi_K)$ converges to infinity. From Lemma 1 and 2 *(ii)* follows that for $\psi > \theta > 0$, $g$ converges to zero. Because of (A11), (A14) and the convergence property of the natural logarithm, $g$ converges faster than the logarithm it is multiplied with. Thus for $\psi > \theta > 0$

$$\lim_{k \to \infty} \frac{\partial w}{\partial \psi} = -\infty [\ln \pi_0 + 0] = \infty. \tag{A32}$$

For $\theta > \psi > 0$ we rewrite

$$\lim_{k \to \infty} \frac{\partial w}{\partial \psi} = \lim_{k \to \infty} \left( \frac{-y}{\psi^2} \right) (1 - \pi_X) \pi_K \left[ \Phi_X \frac{1}{g} + \pi_X \ln \left( \frac{\pi_X (1 - \pi_{X0})}{\pi_{X0}(1 - \pi_X)} \right) \right]. \tag{A33}$$

From Lemma 1 and 2 *(i)* follows that for $\theta > \psi > 0$, $g$ converges to one, thus:

$$\lim_{k \to \infty} \frac{\partial w}{\partial \psi} = -\infty [\ln \pi_0 * 1 + \infty] = -\infty. \blacksquare \tag{A34}$$

Proposition 4 *(iii)* for $\partial w / \partial \theta$ is obtained in an analogous way. With (A2)-(A5) we rewrite (19) as

$$\begin{aligned} \frac{\partial w}{\partial \theta} = -\frac{y}{\theta^2} \pi_X \quad \Bigg[ & ((1 - \pi_K) + \pi_K (1 - \pi_X)(1 - \psi)) \Phi_K \\ + & \theta \pi_K (1 - \pi_K) \ln \left( \frac{\pi_K}{\pi_{K0}} \frac{1 - \pi_{K0}}{1 - \pi_K} \right) \Bigg]. \end{aligned} \tag{A35}$$

For $\psi$ or $\theta < 0$ one sees that $\lim_{k \to \infty} \partial w / \partial \theta > 0$. For $\psi, \theta > 0$ the behavior of $\frac{g}{1-g}$ is considered.$\blacksquare$

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